

Minimal surfaces with limit ends in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

For any $m \geq 1$, we construct properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and m limit ends. We also provide examples with an infinite countable number of limit ends. All these examples are vertical bi-graphs.

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42

1 Introduction

The theory of properly embedded minimal surfaces with genus zero (i.e. those which are topologically a punctured sphere) in Euclidean space \mathbb{R}^3 has been largely studied (see [1, 2, 8, 9] and the references therein). The final classification of such surfaces was given by Bill Meeks, Joaquín Pérez and Antonio Ros in [10]. The only examples with infinite topology are Riemann minimal surfaces. They form a 1-parameter family whose natural limits are the catenoid and the helicoid. When the planar ends are horizontally placed, each Riemann minimal example is invariant by a non-horizontal translation, its intersection with any horizontal plane is either a circle or a straight line, it has infinitely many annular ends asymptotic to horizontal planes, and has exactly two limit ends¹: one top and one bottom limit end.

Laurent Hauswirth [5] constructed Riemann-type minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, which are properly embedded and have genus zero, infinitely many ends asymptotic to horizontal slices and two limit ends: one top and one bottom limit end. It is natural to ask if there are examples of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely

*Research partially supported by a Spanish MEC-FEDER Grant no. MTM2007-61775, a Regional J. Andalucía Grant no. P09-FQM-5088 and "Grupo Singular" of the UCM.

¹A limit end e of a non-compact surface M is an accumulation point of the set $\mathcal{E}(M)$ of ends of M . See [9].

many ends and m_0 limit ends, with $m_0 \neq 2$. In this paper we construct examples for any $m_0 \geq 1$, and we also construct examples with an infinite countable number of limit ends.

In a joint work with Filippo Morabito [11], we have recently constructed a $(2k - 3)$ -parameter family \mathcal{F}_k of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with total (intrinsic) curvature $4\pi(1 - k)$, genus zero and k vertical planar ends (i.e. annular ends asymptotic to vertical geodesic planes), for any $k \geq 2$. We call them *minimal k -noids*. Each surface in this family is invariant by reflection symmetry about the horizontal slice $\mathbb{H}^2 \times \{0\}$. Pyo [13] has constructed independently a 1-parameter family of surfaces with the same properties. The examples given by Pyo, which are included in \mathcal{F}_k , are also invariant by reflection symmetry about k vertical geodesic planes forming an angle π/k . The examples with genus zero and infinitely many ends we construct in the present paper are obtained by taking limits when $k \rightarrow +\infty$ of certain surfaces $M_k \in \mathcal{F}_k$.

The simple ends (i.e. non-limit ends) of the surfaces we construct are vertical planar ends. Fixed an orientation of \mathbb{H}^2 , we can order these simple ends cyclically. If a limit end can be obtained as accumulation of simple ends ordered following the negative orientation (resp. the positive orientation) but it cannot be obtained as accumulation of simple ends ordered following the positive orientation (resp. negative orientation), we will say that it is a *left* (resp. *right*) limit end. In other case, we will say that it is a *2-sided* limit end.

Now we state the main results of this paper.

Theorem 1.1. *For any $m_0 \geq 1$, there exists a properly embedded minimal surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and m_0 limit ends, which is symmetric with respect to a horizontal slice (in fact, it is a vertical bi-graph). Moreover, if we denote by $E_\infty^1, \dots, E_\infty^{m_0}$ the limit ends of Σ , we can prescribe each E_∞^m to be left, right or 2-sided.*

If we take limits of appropriately chosen minimal surfaces in \mathcal{F}_k , we can also obtain properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and an infinite countable number of limit ends.

Theorem 1.2. *There exists a properly embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and an infinite countable number of limit ends $\{E_\infty^m\}_{m \in \mathbb{N}}$, which is symmetric with respect to a horizontal slice (in fact, it is a vertical bi-graph). Moreover, we can prescribe each E_∞^m to be left, right or 2-sided.*

All the examples constructed in Theorems 1.1 and 1.2 are obtained by reflection symmetry about the horizontal slice $\mathbb{H}^2 \times \{0\}$ from a vertical graph contained in $\mathbb{H}^2 \times [0, +\infty)$ whose boundary lies on $\mathbb{H}^2 \times \{0\}$.

The author would like to thank Joaquín Pérez for some helpful conversations.

2 Preliminaries

We consider the half-plane model of \mathbb{H}^2 ,

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\},$$

with the hyperbolic metric $g_{-1} = \frac{1}{y^2}(dx^2 + dy^2)$. We denote by t the coordinate in \mathbb{R} and consider in $\mathbb{H}^2 \times \mathbb{R}$ the usual product metric,

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) + dt^2.$$

Given an open domain $\Omega \subset \mathbb{H}^2$ and a smooth function $u : \Omega \rightarrow \mathbb{R}$, the graph of u is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ when

$$(1) \quad \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0,$$

where all terms are calculated with respect to the metric of \mathbb{H}^2 .

Finally, we denote by $\partial_\infty \mathbb{H}^2$ the infinite boundary of \mathbb{H}^2 , i.e.

$$\partial_\infty \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}.$$

2.1 Flux of a minimal graph along a curve

Let u be a minimal graph defined on a domain $\Omega \subset \mathbb{H}^2$. Assume $\partial\Omega$ is piecewise smooth and u extends continuously to $\bar{\Omega}$ (possibly with infinite values). We define the *flux* of u along a curve $\Gamma \subset \partial\Omega$ as

$$F_u(\Gamma) = \int_\Gamma \left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \eta \right\rangle ds,$$

where η is the outer normal to $\partial\Omega$ in \mathbb{H}^2 and ds is the arc-length of $\partial\Omega$.

In the case $\Gamma \subset \Omega$, we can see Γ in the boundary of different subdomains of Ω , with two possible induced orientations. The flux $F_u(\Gamma)$ of u along Γ is then well-defined up to sign, and $|F_u(\Gamma)|$ is well-defined.

Given an arc $C \subset \mathbb{H}^2$, we will denote by $|C|$ the length of C in \mathbb{H}^2 . The proof of the following result can be found in [12], Lemmae 1 and 2.

Lemma 2.1 ([12]). *Let u be a minimal graph on a domain $\Omega \subset \mathbb{H}^2$.*

- (i) For every subdomain $\Omega' \subset \Omega$ such that $\overline{\Omega'}$ is compact, we have $F_u(\partial\Omega') = 0$.
- (ii) Let C be a piecewise smooth curve contained in the interior of Ω , or a convex curve in $\partial\Omega$ where u extends continuously and takes finite values. If C has finite length, then $|F_u(C)| < |C|$.
- (iii) Let $T \subset \partial\Omega$ be a geodesic arc of finite length such that u diverges to $+\infty$ (resp. $-\infty$) as one approaches T within Ω . Then $F_u(T) = |T|$ (resp. $F_u(T) = -|T|$).

The last statement in Lemma 2.1 admits the following generalization.

Lemma 2.2 ([12]). *For each $n \in \mathbb{N}$, let u_n be a minimal graph on a fixed domain $\Omega \subset \mathbb{H}^2$ which extends continuously to $\overline{\Omega}$, and let T be a geodesic arc of finite length in $\partial\Omega$.*

- (i) *If $\{u_n\}_n$ diverges uniformly to $+\infty$ on compact subsets of T while remaining uniformly bounded in compact subsets of Ω , then $F_{u_n}(T) \rightarrow |T|$.*
- (ii) *If $\{u_n\}_n$ diverges uniformly to $+\infty$ in compact subsets of Ω while remaining uniformly bounded on compact subsets of T , then $F_{u_n}(T) \rightarrow -|T|$.*

2.2 Divergence lines

Let $\Omega \subset \mathbb{H}^2$ be a polygonal domain (i.e. a domain whose edges are geodesic arcs of \mathbb{H}^2) with vertices in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, possibly infinitely many. Given a sequence $\{u_k\}_k$ of minimal graphs defined on Ω , we define its *convergence domain* as

$$\mathcal{B} = \{p \in \Omega \mid \{|\nabla u_k(p)|\}_k \text{ is bounded}\},$$

and the *divergence set* of $\{u_k\}_k$ as

$$\mathcal{D} = \Omega - \mathcal{B}.$$

From Lemma 4.3 in [7], we know that the divergence set \mathcal{D} is composed of geodesic arcs contained in Ω , called *divergence lines*, each one joining two points of $\partial\Omega$ (including the vertices of Ω). The following proposition describes the convergence domain and the divergence set of a sequence of minimal graphs. Its proof can be found in [7], Lemmata 4.2 and 4.3 and Proposition 4.4.

Proposition 2.3 ([7]). *Let $\Omega \subset \mathbb{H}^2$ be a polygonal domain, and $\{u_k\}_k$ a sequence of minimal graphs on Ω . Suppose that \mathcal{D} is a countable set of divergence lines. Then passing to a subsequence we have:*

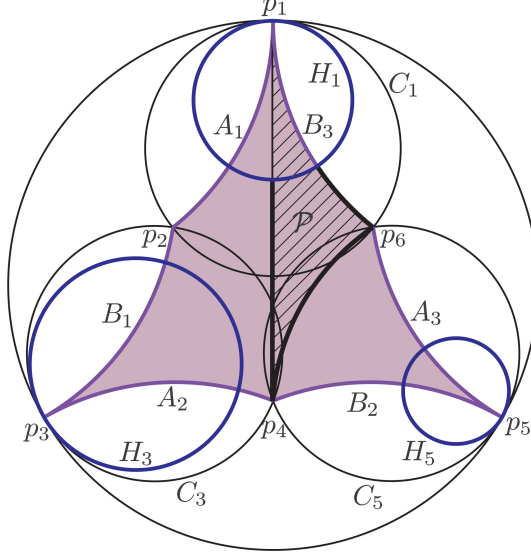


Figure 1: Example of a convex Jenkins-Serrin semi-ideal polygonal domain which verifies condition (\star) .

1. \mathcal{D} is composed of pairwise disjoint geodesic arcs contained in Ω (called divergence lines), each one joining two points in $\partial\Omega$ (including the vertices of Ω).
2. $\{|F_{u_k}(T)|\}_k$ converges to $|T|$ as $k \rightarrow +\infty$, for any geodesic arc T with finite length contained in a divergence line $L \subset \mathcal{D}$.
3. \mathcal{B} is an open set. Moreover, for any component Ω' of \mathcal{B} and any $p \in \Omega'$, $\{u_k - u_k(p)\}_k$ converges uniformly on compact subsets of Ω' to a minimal graph u_∞ .

2.3 Jenkins-Serrin graphs on semi-ideal polygonal domains

Let Ω be a polygonal domain. We say that Ω is *semi-ideal* when no two consecutive vertices of Ω are either in \mathbb{H}^2 or at $\partial_\infty \mathbb{H}^2$ (see Figure 1). We call *interior vertices* of Ω to those which are contained in \mathbb{H}^2 ; *ideal vertices* of Ω to those lying on $\partial_\infty \mathbb{H}^2$; and *limit ideal vertices* of Ω to the limit points of ideal vertices of Ω .

Fix a semi-ideal polygonal domain Ω with a finite number of vertices p_1, \dots, p_{2k} (cyclically ordered). We can assume the odd vertices p_{2i-1} are ideal, and then the even vertices p_{2i} are interior.

For each $i = 1, \dots, k$, we call A_i (resp. B_i) the geodesic arc joining p_{2i-1}, p_{2i} (resp. p_{2i}, p_{2i+1}). We consider a horocycle H_{2i-1} at p_{2i-1} . Assume $H_{2i-1} \cap H_{2j-1} = \emptyset$ for any $i \neq j$. Given a polygonal domain \mathcal{P} inscribed in Ω (i.e. a polygonal domain $\mathcal{P} \subset \Omega$ whose vertices are vertices of Ω , possibly at $\partial_\infty \mathbb{H}^2$), we denote by $\Gamma(\mathcal{P})$ the part of $\partial \mathcal{P}$ outside the horocycles (observe that $\Gamma(\mathcal{P}) = \partial \mathcal{P}$ in the case all the vertices of \mathcal{P} are interior). Also let us call

$$\alpha(\mathcal{P}) = \sum_{i=1}^k |A_i \cap \Gamma(\mathcal{P})| \quad \text{and} \quad \beta(\mathcal{P}) = \sum_{i=1}^k |B_i \cap \Gamma(\mathcal{P})|,$$

where we recall that $|\bullet| = \text{length}_{\mathbb{H}^2}(\bullet)$. See Figure 1.

Definition 2.4. Let Ω be a semi-ideal polygonal domain with a finite number of vertices p_1, \dots, p_{2k} , where $p_{2i-1} \in \partial_\infty \mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$. We say that Ω is *Jenkins-Serrin* if for some choice of horocycles H_{2i-1} as above it holds:

- (i) $\alpha(\Omega) = \beta(\Omega)$.
- (ii) $2\alpha(\mathcal{P}) < |\Gamma(\mathcal{P})|$ and $2\beta(\mathcal{P}) < |\Gamma(\mathcal{P})|$, for every polygonal domain \mathcal{P} inscribed in Ω , $\mathcal{P} \neq \Omega$.

We remark that condition (i) in the above definition does not depend on the choice of horocycles; and if the inequalities of condition (ii) are satisfied for some choice of horocycles, then they continue to hold for “smaller” horocycles (see the argument given by Pascal Collin and Harold Rosenberg in [3], pages 1884 and 1885). The following result is a particular case of Theorem 4.12 in [7].

Theorem 2.5 ([7]). *Let Ω be a semi-ideal polygonal domain with edges $A_1, B_1, \dots, A_k, B_k$ (cyclically ordered). There exists a solution u for the minimal graph equation (1) in Ω with boundary values*

$$u|_{A_i} = +\infty \quad \text{and} \quad u|_{B_i} = -\infty, \quad \text{for any } i = 1, \dots, k,$$

if, and only if, Ω is a Jenkins-Serrin domain. Moreover, such a solution is unique up to an additive constant, when it exists.

We will work with convex semi-ideal polygonal domains Ω satisfying the following additional condition (see Figures 1 and 2):

- (\star) *There exists a choice of pairwise disjoint horocycles H_{2i-1} at the ideal vertices $p_{2i-1} \in \partial_\infty \mathbb{H}^2$ such that*

$$\text{dist}_{\mathbb{H}^2}(p_{2i-2}, H_{2i-1}) = \text{dist}_{\mathbb{H}^2}(p_{2i}, H_{2i-1})$$

for any $i = 1, \dots, k$, using the cyclical notation $p_0 = p_{2k}$.

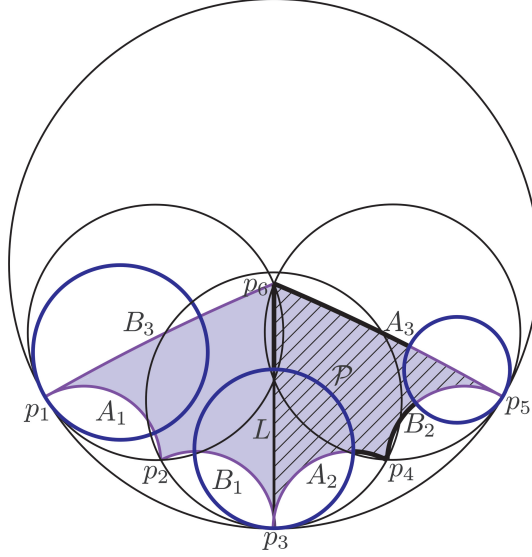


Figure 2: Example of a convex semi-ideal polygonal domain satisfying condition (\star) which is not Jenkins-Serrin, since $2\alpha(\mathcal{P}) > |\Gamma(\mathcal{P})|$ because $|A_2 \cap \Gamma(\mathcal{P})| > |L \cap \Gamma(\mathcal{P})|$.

We remark that condition (\star) does not depend on the choice of horocycles H_{2i-1} , and it is equivalent to the existence of a horocycle C_{2i-1} at p_{2i-1} passing through p_{2i-2}, p_{2i} , for any $i = 1, \dots, k$. We call D_{2i-1} the component of $\mathbb{H}^2 - C_{2i-1}$ whose only point of $\partial_\infty \mathbb{H}^2$ at its infinite boundary is p_{2i-1} (i.e. D_{2i-1} is the horodisk at p_{2i-1} bounded by C_{2i-1}), and $\overline{D_{2i-1}} = D_{2i-1} \cup C_{2i-1}$.

Before finishing this subsection, we describe geometrically when a semi-ideal polygonal domain with a finite number of vertices and satisfying condition (\star) is a Jenkins-Serrin domain. See Figure 2.

Lemma 2.6. *Let Ω be a semi-ideal polygonal domain with vertices p_1, \dots, p_{2k} cyclically ordered so that $p_{2i-1} \in \partial_\infty \mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$, for any $i = 1, \dots, k$. Suppose Ω satisfies condition (\star) above. Then the following assertions are equivalent:*

1. Ω is a Jenkins-Serrin domain.
2. $p_{2j} \in \mathbb{H}^2 - \overline{D_{2i-1}}$, for any j and any $i \notin \{j, j+1\}$.

Proof. Before proving the lemma, let us fix some notation. For any $i = 1, \dots, k$, consider the nested sequence of horocycles $\{H_{2i-1}(n)\}_n$ at p_{2i-1} contained in D_{2i-1} and converging

to p_{2i-1} as $n \rightarrow +\infty$, such that $\text{dist}_{\mathbb{H}^2}(H_{2i-1}(n), C_{2i-1}) = n$, for any n . Then

$$\text{dist}_{\mathbb{H}^2}(p_{2i-2}, H_{2i-1}(n)) = \text{dist}_{\mathbb{H}^2}(p_{2i}, H_{2i}(n)) = n.$$

Let us now prove Lemma 2.6. First suppose Ω is Jenkins-Serrin and there exists some $p_{2j} \in \overline{D_{2i-1}}$, with $i \notin \{j, j+1\}$. We then have

$$\text{dist}_{\mathbb{H}^2}(p_{2j}, H_{2i-1}(n)) \leq n$$

for n large. Let L be the geodesic arc from p_{2j} to p_{2i-1} , and \mathcal{P} be the component of Ω —containing A_i on its boundary (see Figure 2). Clearly, \mathcal{P} is a polygonal domain inscribed in Ω . And, for this choice of horocycles $H_{2i-1}(n)$, it holds $|A_\ell \cap \Gamma(\mathcal{P})| = n$ (resp. $|B_\ell \cap \Gamma(\mathcal{P})| = n$) for any ℓ such that $A_\ell \subset \partial\mathcal{P}$ (resp. $B_\ell \subset \partial\mathcal{P}$). Thus $\beta(\mathcal{P}) = \alpha(\mathcal{P}) - n$, and then

$$|\Gamma(\mathcal{P})| = \text{dist}_{\mathbb{H}^2}(p_{2j}, H_{2i-1}(n)) + \alpha(\mathcal{P}) + \beta(\mathcal{P}) \leq 2\alpha(\mathcal{P}).$$

This holds for every n large, which contradicts that Ω is a Jenkins-Serrin domain. This proves (1) \Rightarrow (2).

Now assume $p_{2j} \in \mathbb{H}^2 - \overline{D_{2i-1}}$, for any j and any $i \notin \{j, j+1\}$, and let us prove that Ω is a Jenkins-Serrin domain. As we have remarked above, we have $\alpha(\Omega) = \beta(\Omega)$. Suppose there exists an inscribed polygonal domain \mathcal{P} in Ω , $\mathcal{P} \neq \Omega$, such that

$$|\Gamma(\mathcal{P})| \leq 2\alpha(\mathcal{P})$$

(the case $|\Gamma(\mathcal{P})| \leq 2\beta(\mathcal{P})$ follows similarly). Since $\mathcal{P} \neq \Omega$, there is at least an interior geodesic γ_1 in $\partial\mathcal{P}$ (i.e. $\gamma_1 \subset \partial\mathcal{P} \cap \Omega$). We can assume there are no two consecutive interior geodesics γ_1, γ_2 in $\partial\mathcal{P}$: We would replace \mathcal{P} by another inscribed polygonal domain satisfying the same properties as \mathcal{P} by replacing $\gamma_1 \cup \gamma_2$ by the geodesic γ_3 such that $\gamma_1 \cup \gamma_2 \cup \gamma_3$ bounds a geodesic triangle contained in Ω . In a similar way, we can assume that

$$\partial\mathcal{P} = A_{i_1} \cup \gamma_1 \cup \dots \cup A_{i_j} \cup \gamma_j \cup A_{i_{j+1}} \cup \dots \cup A_{i_s} \cup \gamma_s,$$

where each γ_j is either an interior geodesic or a B_i edge, and at least $\gamma_1 \subset \Omega$. In particular, each γ_j joins an even vertex p_{2i_j} to an odd vertex $p_{2i_{j+1}-1}$. (Observe that, when γ_j is a B_i edge, then $\gamma_j = B_{i_j}$ and $i_{j+1} = i_j + 1$.)

Hence $\sum_{j=1}^s |\gamma_j \cap \Gamma(\mathcal{P})| = |\Gamma(\mathcal{P})| - \alpha(\mathcal{P}) \leq \alpha(\mathcal{P}) = sn$, from where we deduce there must be an interior geodesic $\gamma_j \subset \partial\mathcal{P}$ whose length is smaller than or equal to n . But this implies the vertex p_{2i_j} lies on $\overline{D_{2i_{j+1}-1}}$ and $i_j \notin \{i_{j+1}-1, i_{j+1}\}$, a contradiction. \square

2.4 Conjugate surfaces in $\mathbb{H}^2 \times \mathbb{R}$

In this subsection we briefly recall how to obtain minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ by conjugation from other known minimal examples. For more details see Daniel [4, Section 4] and Hauswirth, Sa Earp and Toubiana [6].

Let Σ be a 2-sided minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. We call height function of Σ to the horizontal projection $h : \Sigma \rightarrow \mathbb{R}$, which is known to be a real harmonic map. And we denote by $F : \Sigma \rightarrow \mathbb{H}^2$ the vertical projection, which is a harmonic map, and by

$$Q = \langle F_z \bar{F}_z \rangle (dz)^2$$

the Hopf differential associated to F , where z is a local conformal coordinate on Σ . Finally, we denote by N a globally defined unit normal vector field on Σ and by $\nu = \langle N, \frac{\partial}{\partial t} \rangle$ the angle function of Σ .

Theorem 2.7 ([4, 6]). *Let Σ be a simply-connected minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. There exists a minimal surface $\Sigma^* \subset \mathbb{H}^2 \times \mathbb{R}$, called conjugate surface of Σ , such that:*

1. *Σ and Σ^* are isometric. (If we identify points in Σ and Σ^* via an isometry, we can assume that the angle function ν^* , the height function h^* , the vertical projection F^* of Σ^* , and the Hopf differential Q^* associated to F^* , are all defined on Σ .)*
2. *The angle functions ν, ν^* coincide.*
3. *The height functions h, h^* are real harmonic conjugate.*
4. *$Q^* = -Q$.*

The conjugate surface Σ^ is well-defined up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$. Finally, the conjugation exchanges the following Schwarz reflections:*

- *The symmetry with respect to a vertical geodesic plane of $\mathbb{H}^2 \times \mathbb{R}$ containing a geodesic curvature line of Σ becomes the rotation of $\mathbb{H}^2 \times \mathbb{R}$ by angle π with respect to a horizontal geodesic contained in Σ^* , and viceversa.*
- *The symmetry with respect to a horizontal slice containing a geodesic curvature line of Σ becomes the rotation by angle π with respect to a vertical straight line contained in Σ^* , and viceversa.*

We will use the above correspondence to study the conjugate surface of a minimal graph defined on a convex semi-ideal polygonal domain of \mathbb{H}^2 . The surface constructed in this way is a minimal graph (and consequently embedded), as ensured by the following Krust-type theorem.

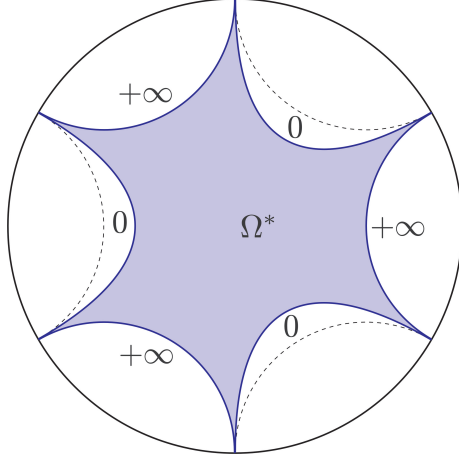


Figure 3: Vertical projection of the conjugate surface Σ^* in a symmetric case.

Theorem 2.8 ([6]). *If Σ is a minimal graph over a convex domain Ω of \mathbb{H}^2 , then Σ^* is also a minimal graph over a (non-necessarily convex) domain $\Omega^* \subset \mathbb{H}^2$.*

2.5 Minimal k -noids of $\mathbb{H}^2 \times \mathbb{R}$

In this subsection we briefly explain the construction of the properly embedded minimal surfaces of $\mathbb{H}^2 \times \mathbb{R}$ given in [11, 13], which have genus zero, $k \geq 2$ vertical planar ends and finite total (intrinsic) curvature $4\pi(1 - k)$. We call them minimal k -noids of $\mathbb{H}^2 \times \mathbb{R}$.

Let Ω be a convex Jenkins-Serrin semi-ideal polygonal domain with $2k$ vertices p_1, \dots, p_{2k} , cyclically ordered, so that the even vertices p_{2i} are located in the interior of \mathbb{H}^2 , and the odd vertices p_{2i-1} are at $\partial_\infty \mathbb{H}^2$, for $i = 1, \dots, k$. We call A_i the edge of Ω whose endpoints are p_{2i-1}, p_{2i} , and B_i the edge of Ω whose endpoints are p_{2i}, p_{2i+1} . We also require that Ω satisfies the condition (\star) defined in Subsection 2.3.

By Theorem 2.5, there exists a unique solution u to the minimal graph equation (1) defined over Ω with boundary values $+\infty$ on A_i and $-\infty$ on B_i such that $u(p_0) = 0$, for some fixed point $p_0 \in \Omega$. Denote by Σ the graph surface of u ; Σ is bounded by the k vertical straight lines $\Gamma_i = \{p_{2i}\} \times \mathbb{R}$, $i = 1, \dots, k$.

The conjugate surface Σ^* of Σ is a minimal graph over a (non-necessarily convex) domain $\Omega^* \subset \mathbb{H}^2$, by Theorem 2.8 (see Figure 3). And $\partial \Sigma^*$ consists of k horizontal geodesic curvature lines Γ_i^* . In [11] it is proved that $\Gamma_i^* \subset \mathbb{H}^2 \times \{0\}$ for any i and that Σ^* is contained in one of the half-spaces determined by $\mathbb{H}^2 \times \{0\}$. By reflecting Σ^* with respect to $\mathbb{H}^2 \times \{0\}$, we get a properly embedded minimal surface M with genus zero and

k vertical planar ends, which has total (intrinsic) curvature $4\pi(1-k)$. The ends of M are asymptotic to the vertical geodesic planes $\eta_i^* \times \mathbb{R}$, where the η_i^* are the complete geodesics such that $\partial\Omega^* = \Gamma_1^* \cup \eta_1^* \cup \dots \cup \Gamma_k^* \cup \eta_k^*$ (cyclically ordered).

3 Proof of Theorem 1.1: examples with m_0 limit ends

Firstly, let us recall some definitions. A limit end e of a non-compact surface M is an accumulation point of the set $\mathcal{E}(M)$ of ends of M . This makes sense since $\mathcal{E}(M)$ can be endowed with a natural topology for which it is a compact, totally disconnected subspace of the real interval $[0, 1]$. See [9] for more details. We call simple ends of M to its non-limit ends.

Assume the simple ends of M are asymptotic to vertical geodesic planes (called vertical planar ends) which can be ordered cyclically², fixed an orientation of $\partial_\infty \mathbb{H}^2$. If the limit end can be obtained as accumulation of simple ends ordered following the negative orientation (resp. the positive orientation) but it cannot be obtained as accumulation of simple ends ordered following the positive orientation (resp. negative orientation), we will say that it is a *left* (resp. *right*) limit end. In other case, we will say that it is a *2-sided* limit end.

In this section we construct properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and m_0 limit ends, for any $m_0 \geq 1$. Furthermore, we can prescribe the behavior of the limit ends; more precisely, if we denote by $E_\infty^1, \dots, E_\infty^{m_0}$ (cyclically ordered) the limit ends of such a surface, we can prescribe if each E_∞^m is either a left, a right or a 2-sided limit end.

Now let us explain our construction. In a first step we will construct, by taking limits of convex Jenkins-Serrin semi-ideal polygonal domains Ω_k with finitely many vertices, a convex semi-ideal polygonal domain Ω_∞ with an infinite countable set \mathcal{S} of ideal vertices and m_0 limit points $p_\infty^1, \dots, p_\infty^{m_0}$ (cyclically ordered) such that, if E_∞^m is prescribed to be a left (resp. a right or a 2-sided) limit end, then p_∞^m is a left (resp. a right or a 2-sided) limit point, see Definition 3.1 below. We will say that such a limit point p_∞^m is a *left* (resp. a *right* or a *2-sided*) *limit ideal vertex* of Ω_∞ .

Definition 3.1. Let \mathcal{S} be a set of points in $\partial_\infty \mathbb{H}^2 = \{y = 0\}$. We will say that $p_\infty \in \mathcal{S}$ is a limit point if every neighborhood of p_∞ in $\partial_\infty \mathbb{H}^2$ contains a point of \mathcal{S} other than p_∞

²The surface M we want to construct will be obtained as a limit of minimal k -noids, and it is got by reflection symmetry from a minimal graph with boundary values $0, +\infty$, alternately. The vertical projection of M will be bounded by strictly concave curves Γ_i^* and geodesic curves η_i^* , disposed alternately and asymptotic at $\partial_\infty \mathbb{H}^2$. The ends of M will be asymptotic to the vertical geodesic planes $\eta_i^* \times \mathbb{R}$, which are cyclically ordered.

itself; i.e. if $p_\infty = (x_\infty, 0)$, then $\mathcal{S} \cap \{y = 0, 0 < |x - x_\infty| < \varepsilon\} \neq \emptyset$ for every $\varepsilon > 0$; or $p_\infty = \infty$ and $\mathcal{S} \cap \{y = 0, |x| > M\} \neq \emptyset$ for every $M > 0$.

A limit point $p_\infty = (x_\infty, 0)$ of \mathcal{S} is said to be a left (resp. a right) limit point if there exists some $\varepsilon > 0$ such that $\mathcal{S} \cap \{-\varepsilon < x - x_\infty < 0\} = \emptyset$ (resp. $\mathcal{S} \cap \{0 < x - x_\infty < \varepsilon\} = \emptyset$); and it is said to be a 2-sided limit point in other case.

If $p_\infty = \infty$ is a limit point of \mathcal{S} , we say that it is a left (resp. a right) limit point if there exists some $M > 0$ such that $\mathcal{S} \cap \{x > M\} = \emptyset$ (resp. $\mathcal{S} \cap \{x < -M\} = \emptyset$); and it is a 2-sided limit point in other case.

Next we will get a Jenkins-Serrin minimal graph Σ over Ω_∞ as a limit of Jenkins-Serrin minimal graphs over the Ω_k domains. Finally, we will prove that the conjugate surface of Σ is a minimal graph $\Sigma^* \subset \mathbb{H}^2 \times [0, +\infty)$ whose boundary, which consists of horizontal geodesic curvature lines, is contained in the horizontal slice $\mathbb{H}^2 \times \{0\}$. The desired surface is obtained from Σ^* by reflection symmetry about $\mathbb{H}^2 \times \{0\}$.

3.1 Construction of the domains

This subsection deals with the construction of the convex semi-ideal polygonal domain Ω_∞ in the argument explained above. We will construct a sequence of convex Jenkins-Serrin semi-ideal polygonal domains Ω_k satisfying condition (\star) defined in Subsection 2.3, each Ω_k with a finite number of vertices, with $\Omega_k \subset \Omega_{k+1}$ for any k , and such that they converge to a domain Ω_∞ in the desired conditions.

Consider m_0 different ideal points in $\partial_\infty \mathbb{H}^2$:

$$p_\infty^1 = \infty, p_\infty^2 = (x_\infty^2, 0), \dots, p_\infty^{m_0} = (x_\infty^{m_0}, 0),$$

with $-\infty < x_\infty^2 < \dots < x_\infty^{m_0} < +\infty$, when $m_0 \geq 2$; in the case $m_0 = 1$, we only have $p_\infty^1 = \infty$. These points will be the limit ideal vertices of Ω_∞ .

We call $\mathcal{M} = \{m \in \mathbb{N} \mid 1 \leq m \leq m_0\}$. For any $m \in \mathcal{M}$, choose two ideal points $p_{-1}^m = (x_{-1}^m, 0)$, $p_1^m = (x_1^m, 0)$ with $x_\infty^m < x_{-1}^m < x_1^m < x_\infty^{m+1}$, where $x_\infty^1 = -\infty$ and $x_\infty^{m_0+1} = +\infty$.

For any $m \in \mathcal{M}$ and any $j \in \{-1, 1, \infty\}$, we call C_j^m the horocycle at p_j^m passing through $P_0 = (0, 1)$. Denote by p_0^m (resp. q_{-2}^m, q_2^m) the point in $C_{-1}^m \cap C_1^m$ (resp. $C_\infty^m \cap C_{-1}^m, C_1^m \cap C_\infty^{m+1}$) which is different from P_0 , see Figure 4. We define Ω_1 as the semi-ideal polygonal domain with set of vertices

$$\{p_\infty^m, q_{-2}^m, p_{-1}^m, p_0^m, p_1^m, q_2^m \mid m \in \mathcal{M}\}.$$

By definition of q_{-2}^m, p_0^m, q_2^m , it is clear that Ω_1 satisfies condition (\star) . Now let us see we can choose the ideal vertices p_{-1}^m, p_1^m to assure Ω_1 is convex. It suffices to choose appropriately x_{-1}^m, x_1^m such that $q_{-2}^m, p_0^m, q_2^m \subset \{0 < y < 1\}$, except for $q_{-2}^1, q_2^{m_0} \in \{y = 1\}$.

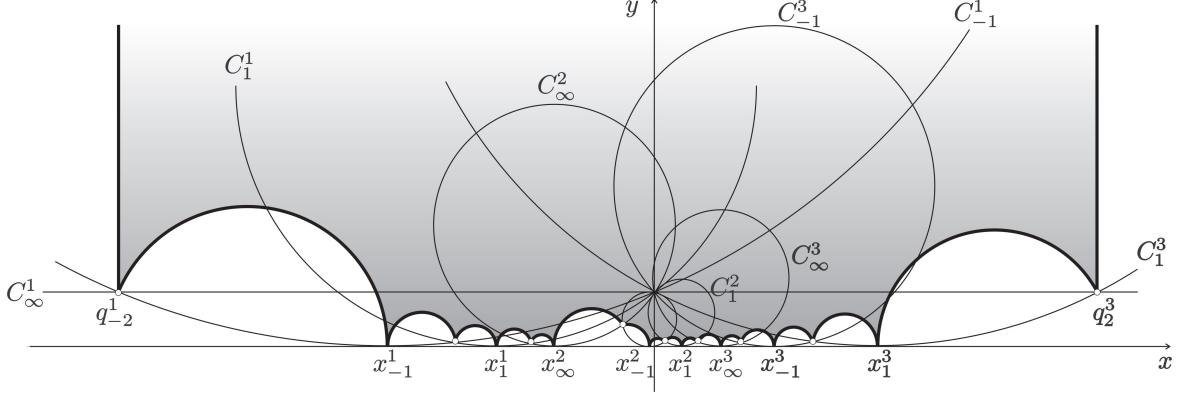


Figure 4: The shadowed region is an example of Ω_1 for three limit ideal vertices, where $p_{\infty}^1 = \infty$ and $p_i^m = (x_i^m, 0)$ for the remaining values of i, m . The interior vertices in white correspond, from the left to the right, to $q_{-2}^1, p_0^1, q_2^1, q_{-2}^2, p_0^2, q_2^2, q_{-2}^3, p_0^3, q_2^3$.

- If $x_{\infty}^m \geq 0$ or $x_{\infty}^{m+1} \leq 0$, then x_{-1}^m, x_1^m can be chosen arbitrarily.
- In the case $x_{\infty}^m < 0 < x_{\infty}^{m+1}$, we take $\max\{x_{\infty}^m, -1\} < x_{-1}^m < 0 < x_1^m < \min\{x_{\infty}^{m+1}, 1\}$.

With the choice above, the domain Ω_1 is convex. Finally, let us check that Ω_1 is a Jenkins-Serrin domain. Using Lemma 2.6 it suffices to get that p_0^m (resp. $q_{-2}^m; q_2^m$) lies outside $C_j^{m'}$, for any $j \in \{-1, 1, \infty\}$ and any $m' \in \mathcal{M}$ such that $C_j^{m'}$ is different from C_{-1}^m, C_1^m (resp. $C_{\infty}^m, C_{-1}^m; C_1^m, C_{\infty}^{m+1}$). By the choice above, this is the case when $C_j^{m'} = C_{\infty}^1$. Let us assume $C_j^{m'} \neq C_{\infty}^1$. We prove it for p_0^m (for q_{-2}^m, q_2^m it can be obtained similarly): If we denote $p_0^m = (x_0^m, y_0^m)$, we have $x_{-1}^m < x_0^m < x_1^m$. If $x_j^{m'} > x_1^m$ (resp. $x_j^{m'} < x_{-1}^m$), then $C_j^{m'}$ divides C_1^m (resp. C_{-1}^m) in two components, one of them containing both p_0^m and p_1^m (resp. p_{-1}^m). That says that p_0^m is outside $C_j^{m'}$.

Now we consider the subsets of \mathcal{M} given by

$$\mathcal{M}^+ = \{m \in \mathcal{M} \mid E_{\infty}^{m+1} \text{ is prescribed to be either a right or a 2-sided limit end}\},$$

$$\mathcal{M}^- = \{m \in \mathcal{M} \mid E_{\infty}^m \text{ is prescribed to be either a left or a 2-sided limit end}\}.$$

For any $k \geq 2$, we define Ω_k as the semi-ideal polygonal domain with set of vertices $\mathcal{V}_k^- \cup \mathcal{V}^0 \cup \mathcal{V}_k^+$, where

$$\mathcal{V}_k^- = \{q_{-2k}^m, p_{1-2k}^m, p_{2-2k}^m, \dots, p_{-3}^m, p_{-2}^m \mid m \in \mathcal{M}^-\} \cup \{q_{-2}^m \mid m \in \mathcal{M} - \mathcal{M}^-\},$$

$$\mathcal{V}_k^0 = \{p_\infty^m, p_{-1}^m, p_0^m, p_1^m \mid m \in \mathcal{M}\},$$

$$\mathcal{V}_k^+ = \{p_2^m, p_3^m, \dots, p_{2k-2}^m, p_{2k-1}^m, q_{2k}^m \mid m \in \mathcal{M}^+\} \cup \{q_2^m \mid m \in \mathcal{M} - \mathcal{M}^+\}.$$

and the vertices $p_{\pm i}^m, q_{\pm 2k}^m$ are defined by induction as follows:

1. Suppose that $m \in \mathcal{M}^+$ and that we have defined the ideal vertices

$$p_1^m = (x_1^m, 0), \dots, p_{2k-1}^m = (x_{2k-1}^m, 0),$$

with $k \geq 1$ and $x_1^m < \dots < x_{2k-1}^m < x_\infty^m$. These ideal vertices determine the following data: For $1 \leq i \leq k$,

- let C_{2i-1}^m be the horocycle at p_{2i-1}^m passing through P_0 ;
- p_{2i-2}^m is defined as the intersection point in $C_{2i-3}^m \cap C_{2i-1}^m$ different from P_0 ;
- q_{2i}^m is the intersection point in $C_{2i-1}^m \cap C_\infty^{m+1}$ different from P_0 .

This choice of p_{2i-2}^m, q_{2i}^m will assure that Ω_k is a convex Jenkins-Serrin semi-ideal polygonal domain which satisfies condition (\star) .

Let us now define p_{2k+1}^m . We call Γ_{2k}^m (resp. γ_{2k}^m) the complete geodesic curve with endpoint p_{2k-1}^m (resp. p_∞^{m+1}) passing through q_{2k}^m . Let $(a_{2k}^m, 0)$ (resp. $(b_{2k}^m, 0)$) be the endpoint of Γ_{2k}^m (resp. γ_{2k}^m) different from p_{2k-1}^m (resp. p_∞^{m+1}). We take $p_{2k+1}^m = (x_{2k+1}^m, 0)$ satisfying $b_{2k}^m \leq x_{2k+1}^m \leq a_{2k}^m$. We consider that property for p_{2k+1}^m in order to get $\Omega_k \subset \Omega_{k+1}$.

We remark that both p_{2k+1}^m and p_{2k}^m converge to p_∞^{m+1} as $k \rightarrow +\infty$.

2. The corresponding definition for $m \in \mathcal{M}^-$ follows analogously: Suppose $m \in \mathcal{M}^-$ and that, for $k \geq 1$, we have defined the ideal vertices

$$p_{1-2k}^m = (x_{1-2k}^m, 0), \dots, p_{-1}^m = (x_{-1}^m, 0),$$

with $x_\infty^m < x_{1-2k}^m < \dots < x_{-1}^m$. These ideal vertices determine the following data: For $1 \leq i \leq k$,

- let C_{1-2i}^m be the horocycle at p_{1-2i}^m passing through P_0 ;
- p_{2-2i}^m is defined as the intersection point in $C_{1-2i}^m \cap C_{3-2i}^m$ different from P_0 ;
- q_{-2i}^m is the intersection point in $C_\infty^m \cap C_{1-2i}^m$ different from P_0 .

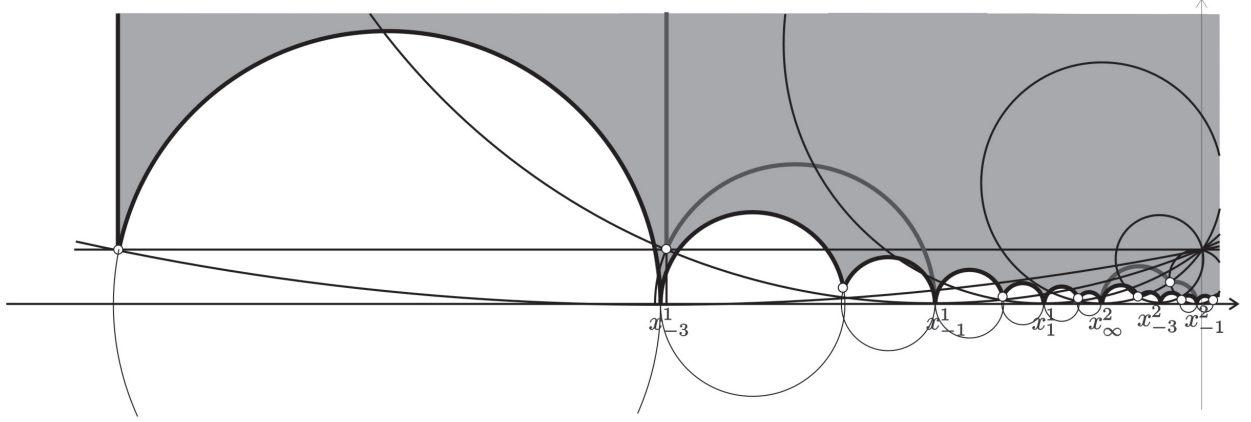


Figure 5: The shadowed region is a piece of Ω_2 , with $1 \in \mathcal{M}^- - \mathcal{M}^+$ and $2 \in \mathcal{M}^-$.

Let us now define p_{-1-2k}^m . We call Γ_{-2k}^m (resp. γ_{-2k}^m) the complete geodesic curve with endpoint p_{-1-2k}^m (resp. p_∞^m) passing through q_{-2k}^m . Let $(a_{-2k}^m, 0)$ (resp. $(b_{-2k}^m, 0)$) be the endpoint of Γ_{-2k}^m (resp. γ_{-2k}^m) different from p_{-1-2k}^m (resp. p_∞^m). We choose $p_{-1-2k}^m = (x_{-1-2k}^m, 0)$, with $b_{-2k}^m \leq x_{-1-2k}^m \leq a_{-2k}^m$. With this choice of ideal vertices, we get $\Omega_k \subset \Omega_{k+1}$ and that both p_{-1-2k}^m and p_{-2k}^m converges to p_∞^m as $k \rightarrow +\infty$.

By definition of the interior vertices p_{2i-2}^m, q_{2i}^m , the semi-ideal polygonal domain Ω_k satisfies condition (\star) . As x_{2k-1}^m has the same sign as x_{2k-3}^m and x_∞^{m+1} , then $p_{2i-2}^m, q_{2i}^m \subset \{0 < y < 1\}$. That fact assures that Ω_k is convex. Moreover, as the horocycles $C_j^{m'}$ can be ordered from the left to the right and they all pass through P_0 , we can deduce (as in the case of Ω_1) that the interior vertices $p_{2i}^m, q_{-2k}^m, q_{2k}^m$ are outside the horocycles $C_j^{m'}$, except for those used for defining them (i.e. their consecutive ones). Then Ω_k is a Jenkins-Serrin domain, by Lemma 2.6.

Finally, we have defined the ideal vertices p_{2i-1}^m to get $\Omega_k \subset \Omega_{k+1}$; for instance, when $m \in \mathcal{M}^+$ and k is positive, the geodesics $A_k^m, B_k^m, \tilde{A}_{k+1}^m$ do not intersect \tilde{A}_k^m , where A_k^m (resp. $B_k^m; \tilde{A}_{k+1}^m; \tilde{A}_k^m$) is defined as the geodesic arc joining p_{2k-1}^m, p_{2k}^m (resp. $p_{2k}^m, p_{2k+1}^m; p_{2k+1}^m, q_{2k+2}^m; p_{2k-1}^m, q_{2k}^m$).

Let Ω_∞ be the semi-ideal polygonal domain with set of vertices $\mathcal{V}_\infty^- \cup \mathcal{V}^0 \cup \mathcal{V}_\infty^+$, where

$$\mathcal{V}_\infty^- = \{p_{-2k}^m, p_{1-2k}^m \mid m \in \mathcal{M}^-, k \in \mathbb{N}\} \cup \{q_{-2}^m \mid m \in \mathcal{M} - \mathcal{M}^-\},$$

$$\mathcal{V}_\infty^+ = \{p_{2k-1}^m, p_{2k}^m \mid m \in \mathcal{M}^+, k \in \mathbb{N}\} \cup \{q_2^m \mid m \in \mathcal{M} - \mathcal{M}^+\}.$$

It is clear that $\Omega_k \rightarrow \Omega_\infty$ as $k \rightarrow +\infty$. We can deduce, arguing as above for Ω_k , that Ω_∞ is a convex semi-ideal polygonal domain verifying condition (\star) (the same condition can be defined for the case of infinitely many vertices).

Since $p_{2k-1} \rightarrow p_\infty^m$ as $k \rightarrow +\infty$ when $m \in \mathcal{M}^+$, and $p_{2k-1} \rightarrow p_\infty^{m-1}$ as $k \rightarrow -\infty$ when $m \in \mathcal{M}^-$, then each p_∞^m is a limit ideal vertex of Ω_∞ and it is:

- left when $m \in \mathcal{M}^-$ and $m-1 \notin \mathcal{M}^+$;
- right when $m \notin \mathcal{M}^-$ and $m-1 \in \mathcal{M}^+$;
- or 2-sided when $m \in \mathcal{M}^-$ and $m-1 \in \mathcal{M}^+$.

3.2 Construction of the Jenkins-Serrin minimal graphs

Let Ω_k, Ω_∞ be the domains constructed above. We call A_i^m (resp. B_i^m) the geodesic arc joining p_{2i-1}^m, p_{2i}^m (resp. p_{2i}^m, p_{2i+1}^m), when they are defined; and \tilde{A}_k^m (resp. $\tilde{B}_k^m; \tilde{A}_{-k}^m; \tilde{B}_{-k}^m$) the geodesic arc joining p_{2k-1}^m, q_{2k}^m (resp. $q_{2k}^m, p_\infty^{m+1}; p_\infty^m, q_{-2k}^m; q_{-2k}^m, p_{1-2k}^m$).

By Theorem 2.5, there exists a solution u_k (unique up to an additive constant) for the minimal graph equation (1) in Ω_k with boundary values $+\infty$ (resp. $-\infty$) on edges $A_i^m, \tilde{A}_k^m, \tilde{A}_{-k}^m$ (resp. $B_i^m, \tilde{B}_k^m, \tilde{B}_{-k}^m$) which lie on $\partial\Omega_k$.

Fix a point $P \in \Omega_1$. We translate vertically the Jenkins-Serrin graphs so that $u_k(P) = 0$, for any k .

Lemma 3.2. *The sequence $\{u_k\}_k$ has no divergence lines.*

Proof. Firstly, let us introduce some notation: For $\mu = 2i-1$ or $\mu = \infty$, consider a sequence of nested horocycles $H_\mu^m(n)$ at p_μ^m (in the case p_μ^m is defined) contained in C_μ^m such that $\text{dist}_{\mathbb{H}^2}(H_\mu^m(n), C_\mu^m) = n$ for any n . In particular, the horocycles $H_{2i-1}^m(n)$ are pairwise disjoint for n large. Given a polygonal domain \mathcal{P}_k inscribed in Ω_k , denote by $\mathcal{P}_k(n)$ the polygonal domain bounded by the part of $\partial\mathcal{P}_k$ outside the horocycles $H_{2i-1}^m(n), H_\infty^m(n)$ together with geodesic arcs joining points in $\partial\mathcal{P}_k \cap ((\cup_{m,i} H_{2i-1}^m(n)) \cup (\cup_m H_\infty^m(n)))$. Also denote

$$\begin{aligned} \alpha_k(n) &= \sum_{m=1}^{m_0} \left(\sum_{i=1-k}^{k-1} |A_i^m \cap \partial\mathcal{P}_k(n)| + |\tilde{A}_{-k}^m \cap \partial\mathcal{P}_k(n)| + |\tilde{A}_k^m \cap \partial\mathcal{P}_k(n)| \right), \\ \beta_k(n) &= \sum_{m=1}^{m_0} \left(\sum_{i=1-k}^{k-1} |B_i^m \cap \partial\mathcal{P}_k(n)| + |\tilde{B}_{-k}^m \cap \partial\mathcal{P}_k(n)| + |\tilde{B}_k^m \cap \partial\mathcal{P}_k(n)| \right), \\ \varepsilon_k(n) &= |\partial\mathcal{P}_k(n) - \partial\mathcal{P}_k|. \end{aligned}$$

We observe that, for any fixed k , $\varepsilon_k(n) \rightarrow 0$ as $n \rightarrow +\infty$.

Now, let us prove Lemma 3.2. Suppose there exists a divergence line L of $\{u_k\}_k$. As $\{\Omega_k\}_k$ is a monotone increasing sequence of domains converging to Ω_∞ , then we can suppose k is large enough so that $L \subset \Omega_k$. We denote by $L(n)$ the geodesic arc in L outside the horocycles $H_{2i-1}^m(n), H_\infty^m(n)$. By Proposition 2.3, $|F_{u_k}(L(n))| \rightarrow |L(n)|$ as $k \rightarrow +\infty$.

We fix a component \mathcal{P}_k of $\Omega_k - L$. By Lemma 2.1,

$$|F_{u_k}(L(n)) + \alpha_k(n) - \beta_k(n)| \leq \varepsilon_k(n),$$

where $\alpha_k(n), \beta_k(n), \varepsilon_k(n)$ are defined as above for this choice of \mathcal{P}_k .

- In the case L has finite length, we have $L(n) = L$ for n large enough. And $\alpha_k(n) - \beta_k(n) = c$ is constant. Taking limits when n goes to $+\infty$, we get $F_{u_k}(L) = -c$. This contradicts the fact that $|F_{u_k}(L)| < |L|$ but $|F_{u_k}(L)| \rightarrow |L|$ as $k \rightarrow +\infty$. Then L must have infinite length.
- If L joins either two ideal vertices $p_{2i-1}^m, p_{2j-1}^{m'}$, two limit ideal vertices $p_\infty^m, p_\infty^{m'}$ or an ideal vertex p_{2i-1}^m to a limit ideal vertex $p_\infty^{m'}$, then we have $\alpha_k(n) = \beta_k(n)$ because of the choice of horocycles above. For any compact geodesic arc $T \subset L(n)$ and k large, we have $|F_{u_k}(T)| \leq |F_{u_k}(L(n))| \leq \varepsilon_k(n)$. Taking $n \rightarrow +\infty$, we get $F_{u_k}(T) = 0$. But this contradicts $|F_{u_k}(T)| \rightarrow |T|$ as $k \rightarrow +\infty$.
- Then L must join a vertex p_μ^m , with $\mu = 2i - 1$ or $\mu = \infty$, to a point q in $\partial\Omega_k \cap \mathbb{H}^2$. Either q is an interior vertex, and we denote it by \tilde{q} , either it lies on an edge of Ω_k and we call \tilde{q} the interior endpoint of such an edge. We can choose \mathcal{P}_k to have $\beta_k(n) \geq \alpha_k(n)$. Hence for n large enough we have that $\beta_k(n) - \alpha_k(n) = n - c$, where $c = \text{dist}_{\mathbb{H}^2}(q, \tilde{q})$. Then

$$F_{u_k}(L(n)) = n - c - F_{u_k}(\partial\mathcal{P}_k(n) - \partial\mathcal{P}_k).$$

(Observe that, in the case L finishes at p_∞^m , all the vertices q_{2i}^m are contained in the same horocycle C_∞^m .) Since $|L(n)| - n = d$ is constant for n large, we get $|L(n)| - |F_{u_k}(L(n))| \rightarrow d + c$ as $n \rightarrow +\infty$. On the other hand, $|F_{u_k}(L(n))| \rightarrow |L(n)|$ as $k \rightarrow +\infty$. So it must hold $c + d = 0$. Therefore,

$$\text{dist}_{\mathbb{H}^2}(\tilde{q}, \partial H_\mu^m(n)) \leq \text{dist}_{\mathbb{H}^2}(\tilde{q}, q) + \text{dist}_{\mathbb{H}^2}(q, \partial H_\mu^m(n)) = c + |L(n)| = n,$$

which implies that \tilde{q} is contained in the horodisk bounded by C_μ^m , in contradiction with the fact that Ω_k is a Jenkins-Serrin domain (see Lemma 2.6).

□

Proposition 3.3. *Passing to a subsequence, $\{u_k\}_k$ converges uniformly on compact subsets of Ω_∞ to a minimal graph u_∞ such that it goes to $+\infty$ (resp. $-\infty$) as we approach within Ω_∞ to each A_i^m and each \tilde{A}_{-1}^m (resp. each B_i^m and each \tilde{B}_1^m) in the boundary of Ω_∞ .*

Proof. Since we have translated vertically the graphs u_k so that $u_k(P) = 0$ for any k , we get from Lemma 3.2 and Proposition 2.3 that, after passing to a subsequence, $\{u_k\}_k$ converges to a minimal graph u_∞ , and the convergence is uniform on compact subsets of Ω_∞ . It is clear that $u_\infty(P) = 0$.

For any bounded geodesic arc $T \subset A_i^m$ we have $F_{u_k}(T) = |T|$ by Proposition 2.3; and then $F_{u_\infty}(T) = |T|$. Hence u_∞ goes to $+\infty$ as we approach T within Ω_∞ . This proves $u_\infty|_{A_i^m} = +\infty$. Similarly we get $u_\infty|_{\tilde{A}_{-1}^m} = +\infty$, $u_\infty|_{B_i^m} = -\infty$ and $u_\infty|_{\tilde{B}_1^m} = -\infty$, which finishes Proposition 3.3. □

3.3 Passing to the conjugate surface

Denote by Σ_∞ (resp. Σ_k) the graph surface of u_∞ (resp. u_k). Observe that, if $m \in \mathcal{M}^+$ (resp. $m \in \mathcal{M}^-$) and $i \geq 1$ (resp. $i \leq -1$), then the vertical straight line $\Gamma_i^m = \{p_{2i}^m\} \times \mathbb{R}$ is contained in the boundary of Σ_∞ and of Σ_k , for any k large; and $\Gamma_0^m = \{p_0^m\} \times \mathbb{R} \subset \partial\Sigma_\infty \cap \partial\Sigma_k$, for any m and any k . We also denote $\tilde{\Gamma}_i^m = \{q_{2i}^m\} \times \mathbb{R}$. Then $\tilde{\Gamma}_k^m \subset \partial\Sigma_k$ and $\tilde{\Gamma}_{-1}^m \subset \partial\Sigma_\infty$ when $m \in \mathcal{M}^+$; and $\tilde{\Gamma}_{-k}^m \subset \partial\Sigma_k$ and $\tilde{\Gamma}_1^m \subset \partial\Sigma_\infty$, when $m \in \mathcal{M}^-$.

We call Σ_k^* the conjugate surface of Σ_k . If Γ is a curve in $\partial\Sigma_k$, then we denote by $\Gamma(k)^*$ the corresponding curve in Σ_k^* . We know (see Subsection 2.5; or [11], section 4) that Σ_k^* is a minimal graph bounded by horizontal geodesic curvature lines contained in the same horizontal slice,

$$\partial\Sigma_k^* = \cup_{m \in \mathcal{M}} (\Upsilon_k^{m-} \cup \Gamma_0^m(k)^* \cup \Upsilon_k^{m+}),$$

where

$$\begin{aligned} \Upsilon_k^{m-} &= \begin{cases} \tilde{\Gamma}_{-k}^m(k)^* \cup \Gamma_{1-k}^m(k)^* \cup \Gamma_{2-k}^m(k)^* \cup \dots \cup \Gamma_{-1}^m(k)^* & , \text{ if } m \in \mathcal{M}^- \\ \tilde{\Gamma}_{-1}^m(k)^* & , \text{ if } m \in \mathcal{M} - \mathcal{M}^- \end{cases} \\ \Upsilon_k^{m+} &= \begin{cases} \Gamma_1^m(k)^* \cup \dots \cup \Gamma_{k-2}^m(k)^* \cup \Gamma_{k-1}^m(k)^* \cup \tilde{\Gamma}_k^m(k)^* & , \text{ if } m \in \mathcal{M}^+ \\ \tilde{\Gamma}_1^m(k)^* & , \text{ if } m \in \mathcal{M} - \mathcal{M}^+ \end{cases} \end{aligned}$$

Up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$, we can assume that the horizontal geodesic curvature lines $\Gamma_i^m(k)^*$ are contained in the horizontal slice $\mathbb{H}^2 \times \{0\}$, and $\Sigma_k^* \subset \{t \geq 0\}$. Each $\Gamma_i^m(k)^*, \tilde{\Gamma}_i^m(k)^* \subset \partial\Sigma_k^*$ corresponds by conjugation, respectively, to $\Gamma_i^m, \tilde{\Gamma}_i^m \subset \partial\Sigma_k$.

If we denote by Ω_k^* the vertical projection of Σ_k^* over $\mathbb{H}^2 \equiv \mathbb{H}^2 \times \{0\}$, then

$$\partial\Omega_k^* = \cup_{m=1}^{m_0} \left(\Lambda_k^{m-} \cup \Gamma_0^m(k)^* \cup \eta_0^m(k)^* \cup \Lambda_k^{m+} \right),$$

cyclically ordered, where

$$\Lambda_k^{m-} = \begin{cases} \tilde{\Gamma}_{-k}^m(k)^* \cup \eta_{-k}^m(k)^* \cup (\cup_{i=1-k}^{-1} (\Gamma_i^m(k)^* \cup \eta_i^m(k)^*)) & , \text{ if } m \in \mathcal{M}^- \\ \tilde{\Gamma}_{-1}^m(k)^* \cup \eta_{-1}^m(k)^* & , \text{ if } m \in \mathcal{M} - \mathcal{M}^- \end{cases}$$

$$\Lambda_k^{m+} = \begin{cases} (\cup_{i=1}^{k-1} (\Gamma_i^m(k)^* \cup \eta_i^m(k)^*)) \cup \tilde{\Gamma}_k^m(k)^* \cup \eta_k^m(k)^* & , \text{ if } m \in \mathcal{M}^+ \\ \tilde{\Gamma}_1^m(k)^* \cup \eta_1^m(k)^* & , \text{ if } m \in \mathcal{M} - \mathcal{M}^+ \end{cases}$$

and each $\eta_i^m(k)^*$ denotes a complete geodesic curve joining at $\partial_\infty \mathbb{H}^2$ the corresponding curves in $\partial\Sigma_k^*$. Furthermore, the curves $\Gamma_i^m(k)^*, \tilde{\Gamma}_i^m(k)^*$ are strictly concave with respect to Ω_k^* (by the maximum principle).

Similarly, we denote by Σ_∞^* the conjugate surface of Σ_∞ .

Proposition 3.4. $\Sigma_\infty^* \subset \{t \geq 0\}$ is a minimal graph over a domain $\Omega_\infty^* \subset \mathbb{H}^2$. Moreover, $\partial\Sigma_\infty^* \subset \mathbb{H}^2 \times \{0\}$ consists of a collection of geodesic curvature lines,

$$\partial\Sigma_\infty^* = \cup_{m=1}^{m_0} \left(\Upsilon_\infty^{m-} \cup \Gamma_0^{m*} \cup \Upsilon_\infty^{m+} \right)$$

(cyclically ordered), where

$$\Upsilon_\infty^{m-} = \begin{cases} \cup_{i=-\infty}^{-1} \Gamma_i^{m*} & , \text{ if } m \in \mathcal{M}^- \\ \widetilde{\Gamma}_{-1}^{m*} & , \text{ if } m \in \mathcal{M} - \mathcal{M}^- \end{cases}$$

$$\Upsilon_\infty^{m+} = \begin{cases} \cup_{i=1}^{+\infty} \Gamma_i^{m*} & , \text{ if } m \in \mathcal{M}^+ \\ \widetilde{\Gamma}_1^{m*} & , \text{ if } m \in \mathcal{M} - \mathcal{M}^+ \end{cases}$$

Each Γ_i^{m*} is strictly concave with respect to Ω_∞^* . Moreover,

$$\partial\Omega_\infty^* = \cup_{m=1}^{m_0} \left(\Lambda_\infty^{m-} \cup \Gamma_0^m \cup \eta_0^{m*} \cup \Lambda_\infty^{m+} \right)$$

(cyclically ordered), with

$$\Lambda_{\infty}^{m-} = \begin{cases} \cup_{i=-\infty}^{-1} (\Gamma_i^{m*} \cup \eta_i^{m*}) & , \text{ if } m \in \mathcal{M}^- \\ \widetilde{\Gamma_{-1}^{m*}} \cup \eta_{-1}^{m*} & , \text{ if } m \in \mathcal{M} - \mathcal{M}^- \end{cases}$$

$$\Lambda_{\infty}^{m+} = \begin{cases} \cup_{i=1}^{+\infty} (\Gamma_i^{m*} \cup \eta_i^{m*}) & , \text{ if } m \in \mathcal{M}^+ \\ \widetilde{\Gamma_1^{m*}} \cup \eta_1^{m*} & , \text{ if } m \in \mathcal{M} - \mathcal{M}^+ \end{cases}$$

where η_i^{m*} denotes a complete geodesic curve asymptotic to its consecutive curves of $\partial\Omega_{\infty}^*$ at $\partial_{\infty}\mathbb{H}^2$.

Proof. Theorem 2.8 says Σ_{∞}^* is a minimal graph over certain domain $\Omega_{\infty}^* \subset \mathbb{H}^2$, because Σ_{∞} is a minimal graph over a convex domain. Moreover, since $\partial\Sigma_{\infty} = \cup_{m,i} \Gamma_i^m$ and each Γ_i^m is a vertical geodesic curve, we get by Theorem 2.7 that the boundary of Σ_{∞}^* is composed of horizontal geodesic curvature lines Γ_i^{m*} . But we do not know a priori if they are all contained in the same horizontal slice.

Let us prove that Σ_{∞}^* can be obtained as a limit of a subsequence of the conjugate graphs Σ_k^* , when k goes to $+\infty$ (in which case the curves $\Gamma_i^m(k)^* \subset \partial\Sigma_k^*$ converge to Γ_i^{m*}). This holds by [11, Proposition 2.10], but we give the idea of the proof: If we prove that, after passing to a subsequence, the graphs Σ_k^* converge to a surface S , then up to isometries of $\mathbb{H}^2 \times \mathbb{R}$ we get $S = \Sigma_{\infty}^*$ by Theorem 6 in [6] (both Σ_{∞}^*, S are isometric to Σ_{∞} ; and the Hopf differentials associated to their vertical projection coincide with $-Q_{\infty}$, where Q_{∞} is the Hopf differential associated to the vertical projection of Σ_{∞}). So we only have to obtain that the sequence $\{\Sigma_k^*\}$ converges. We know that the convergence domain associated to $\{u_k\}_k$ coincides with Ω_{∞} . Then, if we denote by ν_k the angle function of Σ_k , then $\{\nu_k\}_k$ is uniformly bounded away from zero on compact subsets. Since the angle function ν_k^* of Σ_k^* coincides with the one of Σ_k , then the same happens for $\{\nu_k^*\}_k$. We deduce from here that there are no divergence lines for $\{u_k^*\}_k$, and we get the convergence of the graphs Σ_k^* , passing to a subsequence.

Since the graphs Σ_k^* converge to Σ_{∞}^* , then $\Sigma_{\infty}^* \subset \{t \geq 0\}$ and $\partial\Sigma_{\infty}^* \subset \{t = 0\}$. We also deduce that the curves Γ_i^{m*} are cyclically ordered as follows: $\Gamma_i^{m*} \leq \Gamma_j^{m'*}$ if, and only if, $m < m'$ or $m = m'$ and $i \leq j$. By the maximum principle (using vertical geodesic planes), each $\Gamma_i^{m*} \subset \partial\Omega_{\infty}^*$ is strictly concave with respect to Ω_{∞}^* .

Let us now prove that $\Gamma_i^{m*}, \Gamma_{i+1}^{m*}$ cannot finish at the same point Q of $\partial_{\infty}\mathbb{H}^2$. Suppose this is the case. Since $\Gamma_i^{m*}, \Gamma_{i+1}^{m*}$ are strictly concave with respect to Ω_{∞}^* , we get $\text{dist}_{\mathbb{H}^2}(\Gamma_i^{m*}, \Gamma_{i+1}^{m*}) = 0$. Consider a triangle $T \subset \Omega_{\infty}^*$ bounded by subarcs of $\Gamma_i^{m*}, \Gamma_{i+1}^{m*}$ and a geodesic arc c' joining points in $\Gamma_i^{m*}, \Gamma_{i+1}^{m*}$. Let $u_{\infty}^* : T \rightarrow \mathbb{R}$ define the graph

Σ_∞^* over T . Then u_∞^* has boundary values 0 on $\Gamma_i^{m*}, \Gamma_{i+1}^{m*}$ and a bounded continuous function over c' . We call c the complete geodesic of \mathbb{H}^2 containing c' and we consider the minimal graph w^+ (resp. w^-) over the component Δ of $\mathbb{H}^2 - c$ which contains T , which has boundary values $+\infty$ (resp. $-\infty$) over c and 0 over $\partial\Delta \cap \partial_\infty\mathbb{H}^2$. By the maximum principle, $w^-|_T \leq u_\infty^* \leq w^+|_T$. Hence we deduce that u_∞^* converges to 0 as we approach Q in any direction, and then $\text{dist}_{\Sigma_\infty^*}(\Gamma_i^{m*}, \Gamma_{i+1}^{m*}) = 0$. But $\Sigma_\infty, \Sigma_\infty^*$ are isometric and $\text{dist}_{\Sigma_\infty}(\Gamma_i^m, \Gamma_{i+1}^m) \geq \text{dist}_{\mathbb{H}^2}(p_{2i}^m, p_{2i+2}^m) > 0$, a contradiction.

Therefore, the geodesics $\eta_i^m(k)^*$ in the boundary of Ω_k^* converge to a geodesic $\eta_i^{m*} \subset \partial\Omega_\infty^*$ over which Σ_∞^* goes to $+\infty$. Thus $\partial\Omega_\infty^* = \cup_{m=1}^{m_0} \left(\cup_{i=-\infty}^{+\infty} (\Gamma_i^{m*} \cup \eta_i^{m*}) \right)$, cyclically ordered. This finishes the proof of Proposition 3.4. \square

If we reflect Σ_∞^* with respect to $\mathbb{H}^2 \times \{0\}$, we get a properly embedded minimal surface M of genus zero and infinitely many planar ends in $\mathbb{H}^2 \times \mathbb{R}$. The non-limit ends of M are asymptotic to the vertical geodesic planes $\eta_i^{m*} \times \mathbb{R}$. We can deduce that there is exactly one limit end from $\eta_0^{m*} \times \mathbb{R}$ to $\eta_0^{m+1*} \times \mathbb{R}$, that we call E_∞^m ; and E_∞^m is a left (resp. right, 2-sided) limit end when p_∞^m is a left (resp. right, 2-sided) limit ideal vertex.

4 Proof of Theorem 1.1: infinite countable case

In this section we construct properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with genus zero, infinitely many vertical planar ends and an infinite countable number of limit ends $\{E_\infty^m \mid m \in \mathbb{N}\}$. Furthermore, as in the finite case, we can prescribe if each limit end E_∞^k is left, right or 2-sided.

We follow the same sketch as in section 3. We firstly construct, by taking limits of a monotone increasing sequence of convex Jenkins-Serrin semi-ideal polygonal domains Ω_k with finitely many vertices and satisfying condition (\star) , a convex semi-ideal polygonal domain Ω_∞ with an infinite countable number of limit ideal vertices $\{p_\infty^m \mid m \in \mathbb{N}\}$ such that, if E_∞^m is prescribed to be a left (resp. a right or a 2-sided) limit end, then p_∞^m is a left (resp. a right or a 2-sided) limit ideal vertex. The remaining part of the construction follows exactly as in Section 3, replacing m_0 by $+\infty$ and \mathcal{M} by \mathbb{N} .

4.1 Construction of the domains

Consider $p_\infty^1 = \infty$ and two ideal points $p_\infty^2 = (x_\infty^2, 0)$ and $p_\infty^3 = (x_\infty^3, 0)$, with $-1 < x_\infty^2 < x_\infty^3 \leq 1$. These points will be limit ideal vertices of Ω_∞ . We call $C_\infty^1 = \{y = 1\}$ and C_∞^2 (resp. C_∞^3) the horocycle at p_∞^2 (resp. p_∞^3) passing through $P_0 = (0, 1)$.

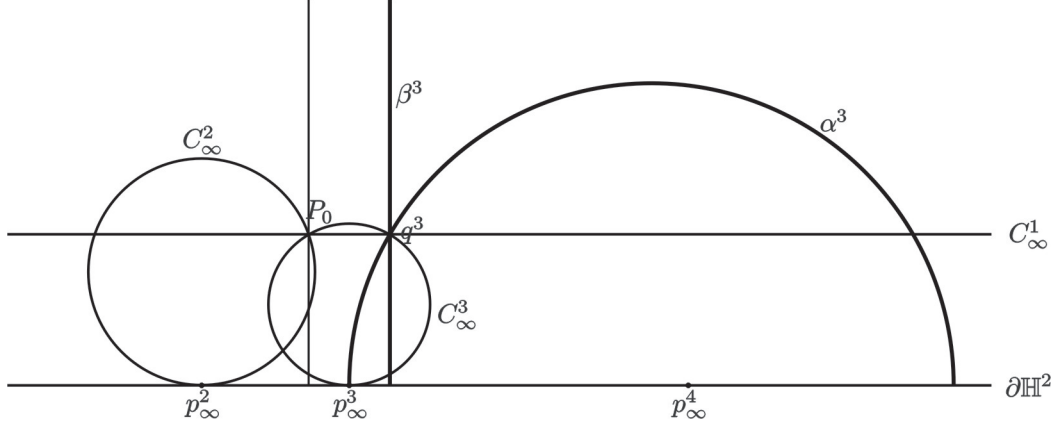


Figure 6: Choice of p_∞^4 .

Let q^3 be the point in $C_\infty^3 \cap C_\infty^1$ different from P_0 , α^3 be the complete geodesic curve with endpoint p_∞^3 passing through q^3 and $\beta^3 = \{x = b^3\}$, where b^3 is the constant for which β^3 passes through q^3 . We take any point $p_\infty^4 = (x_\infty^4, 0)$ with $b^3 \leq x_\infty^4 \leq a^3$, where $(a^3, 0)$ is the endpoint of α^3 different from p_∞^3 , see Figure 6.

Let us now define by induction the remaining limit ideal vertices p_∞^k , $k \in \mathbb{N}$, of Ω_∞ . Assume we have define $p_\infty^4 = (x_\infty^4, 0), \dots, p_\infty^k = (x_\infty^k, 0)$, with $x_\infty^3 < x_\infty^4 < \dots < x_\infty^k < +\infty$. For any $4 \leq i \leq k$, let C_∞^i be the horocycle at p_∞^i passing through $P_0 = (0, 1)$, and q^i be the point in $C_\infty^i \cap C_\infty^1$ different from P_0 . We also consider the complete geodesic curve α^i with endpoint p_∞^i passing through q^i and $\beta^i = \{x = b^i\}$ the geodesic which contains q^i . Denote by $(a^i, 0)$ the endpoint of α^i different from p_∞^i . We assume that $b^{i-1} \leq x_\infty^i \leq a^{i-1}$. We now define p_∞^{k+1} with the same property: We take any point $p_\infty^{k+1} = (x_\infty^{k+1}, 0)$ such that $b^k \leq x_\infty^{k+1} \leq a^k$.

We now want to define the non-limit ideal vertices of Ω_∞ . We consider:

$$\mathcal{M}^+ = \{m \in \mathbb{N} \mid E_\infty^{m+1} \text{ is prescribed to be either a right or a 2-sided limit end}\},$$

$$\mathcal{M}^- = \{m \in \mathbb{N} \mid E_\infty^m \text{ is prescribed to be either a left or a 2-sided limit end}\}.$$

For any $m \in \mathbb{N}$, we define exactly as in Subsection 3.1 (using this new definition of the sets $\mathcal{M}^+, \mathcal{M}^-$) the ideal vertices p_{2i-1}^m of Ω_∞ placed from p_∞^m to p_∞^{m+1} , the horocycles C_{2i-1}^m , the interior vertices p_{2i}^m and the interior points q_{2i}^m .

We can now define the monotone sequence of semi-ideal Jenkins-Serrin polygonal domains Ω_k : We call Ω_1 the semi-ideal polygonal domain with vertices

$$\{p_\infty^m, q_{-2}^m, p_{-1}^m, p_0^m, p_1^m, q_2^m \mid 1 \leq m \leq 2\} \cup \{p_\infty^3, q^3\}.$$

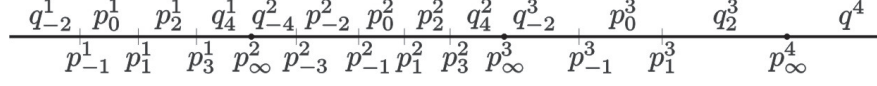


Figure 7: Sketch of the vertices of Ω_2 in the case p_∞^1 and p_∞^3 are right limit ideal vertices, p_∞^2 is a 2-sided limit ideal vertex and p_∞^4 is a left limit ideal vertex .

For $k \geq 2$, let Ω_k be defined as the semi-ideal polygonal domain with vertices

$$\{p_\infty^m, p_{-1}^m, p_0^m, p_1^m \mid 1 \leq m \leq k+1\} \cup \mathcal{V}_k^- \cup \mathcal{V}_k^+ \cup \{p_\infty^{k+2}, q^{k+2}\},$$

where

$$\begin{aligned} \mathcal{V}_k^- &= \{q_{-2k}^m, p_{1-2k}^m, p_{2-2k}^m, \dots, p_{-3}^m, p_{-2}^m \mid 1 \leq m \leq k+1, m \in \mathcal{M}^-\} \\ &\quad \cup \{q_{-2}^m \mid 1 \leq m \leq k+1, m \in \mathcal{M} - \mathcal{M}^-\}, \\ \mathcal{V}_k^+ &= \{p_2^m, p_3^m, \dots, p_{2k-2}^m, p_{2k-1}^m, q_{2k}^m \mid 1 \leq m \leq k+1, m \in \mathcal{M}^+\} \\ &\quad \cup \{q_2^m \mid 1 \leq m \leq k+1, m \in \mathcal{M} - \mathcal{M}^+\}. \end{aligned}$$

The domain Ω_k has $4(k+1) + 2 + N$ vertices, where $2(k+1) \leq N \leq 2(k+1)(2k-1)$ depends on the number of left, right or 2-sided limit ideal ends in $\{E_\infty^1, \dots, E_\infty^{k+1}\}$. As in Subsection 3.1, Ω_k is a convex Jenkins-Serrin semi-ideal polygonal domain satisfying condition (\star) , and $\Omega_k \subset \Omega_{k+1}$. When k goes to $+\infty$, Ω_k converges to the convex semi-ideal polygonal domain Ω_∞ with set of vertices $\mathcal{V}_\infty^- \cup \mathcal{V}_\infty^0 \cup \mathcal{V}_\infty^+$, where

$$\begin{aligned} \mathcal{V}_\infty^- &= \{p_{-2k}^m, p_{1-2k}^m \mid k \in \mathbb{N}, m \in \mathcal{M}^-\} \cup \{q_{-2}^m \mid m \in \mathcal{M} - \mathcal{M}^-\}, \\ \mathcal{V}_\infty^0 &= \{p_\infty^m, p_{-1}^m, p_0^m, p_1^m \mid m \in \mathbb{N}\} \\ \mathcal{V}_\infty^+ &= \{p_{2k-1}^m, p_{2k}^m \mid k \in \mathbb{N}, m \in \mathcal{M}^+\} \cup \{q_2^m \mid m \in \mathcal{M} - \mathcal{M}^+\}. \end{aligned}$$

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